

Semigroups related to additive and multiplicative, free and Boolean convolutions

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Abstract

Belinschi and Nica introduced a composition semigroup on the set of probability measures. Using this semigroup, they introduced a free divisibility indicator, from which one can know whether a probability measure is freely infinitely divisible or not.

In this paper we further investigate this indicator, introduce a multiplicative version of it and are able to show many properties. Specifically, on the first half of the paper, we calculate how the indicator changes with respect to free and Boolean powers; we prove that free and Boolean $1/2$ -stable laws have free divisibility indicators equal to infinity; we derive an upper bound of the indicator in terms of Jacobi parameters. This upper bound is achieved only by free Meixner distributions. We also prove Bożejko's conjecture which says the Boolean power of μ by $t \in [0, 1]$ is freely infinitely divisible if μ is so.

In the other half of this paper, we introduce an analogous composition semigroup for multiplicative convolutions and define free divisibility indicators for these convolutions. Moreover, we prove that a probability measure on the unit circle is freely infinitely divisible concerning the multiplicative free convolution if and only if the indicator is not less than one. We also prove how the multiplicative divisibility indicator changes under free and Boolean powers and then the multiplicative analogue of Bożejko's conjecture. We include an appendix, where the Cauchy distributions and point measures are shown to be the only fixed points of the Boolean-to-free Bercovici-Pata bijection.

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1 Introduction

For probability measures μ, ν on \mathbb{R} , let us denote by $\mu \boxplus \nu$ the free additive convolution. $\mu \boxplus \nu$ is the distribution of $X + Y$, where X and Y are self-adjoint free random variables with distributions μ and ν , respectively. A probability measure μ is said to be freely infinitely divisible if for any $n \in \mathbb{N}$, there exists μ_n such that $\mu = \mu_n \boxplus \cdots \boxplus \mu_n = \mu_n^{\boxplus n}$, the free convolution of μ_n by n times. It is known that for any $t \geq 1$ and any probability measure μ , we can define $\mu^{\boxplus t}$ as a probability measure. This is in contrast to probability theory, because the usual convolution μ^{*t} is not necessarily defined even for $t \geq 1$, unless μ is $*$ -infinitely divisible; the reader is referred to page 2 of [29]. Then the quantity $\tilde{\phi}(\mu) := \inf\{t > 0 : \mu^{\boxplus t} \text{ exists as a probability measure}\}$ is a natural object in free probability. A probability measure μ is freely infinitely divisible if and only if $\tilde{\phi}(\mu) = 0$.

A Boolean additive convolution $\mu \uplus \nu$ is defined as the probability distribution of $X + Y$ for self-adjoint Boolean independent X and Y with distributions μ and ν , respectively. It is known that the power $\mu^{\uplus t}$ can be defined for any probability measure μ and any $t \geq 0$.

Belinschi and Nica [9] introduced a semigroup of homomorphisms using free and Boolean powers. This enables us to define a so-called free divisibility indicator $\phi(\mu)$ of a probability measure μ . They proved $\phi(\mu) = 1 - \tilde{\phi}(\mu)$ if μ is not freely infinitely divisible. A key to the semigroup property is a certain commutation relation between free powers and Boolean ones.

The explicit calculation of this indicator is expected to be useful to understand the free convolution and free infinite divisibility. In that direction, we prove the relation

$$\phi(\mu^{\uplus t}) = \frac{\phi(\mu)}{t} \quad \text{for } t > 0.$$

As a byproduct, we find a different characterization of that indicator in terms of Boolean convolution powers: $\phi(\mu) = \sup\{t \geq 0 : \mu^{\uplus t} \text{ is freely infinitely divisible}\}$. This new characterization enables us to interpret the indicator quite naturally in terms of Boolean convolutions.

Another consequence of this relation is that $\mu^{\uplus t}$ is freely infinitely divisible for $0 \leq t \leq 1$ whenever μ is freely infinitely divisible. This has been conjectured by Bożejko from many calculations [18].

Also, we give an upper bound of the indicator in terms of Jacobi parameters for any probability measure with a finite fourth moment. Moreover, we show that only free Meixner distributions achieve the upper bound.

Delta and Cauchy distributions have free divisibility indicators equal to infinity. This is because they are the fixed points of the semigroup of homomorphisms [9]. They have been the only known examples whose free divisibility indicator is infinity. We will provide other ones: free stable laws and Boolean strictly stable laws of index $1/2$. However, these

examples are not fixed points of the semigroup of homomorphisms. To get these results, we prove that the free divisibility indicator is invariant under shifts and the Boolean convolution with a delta measure. Moreover, we prove in Appendix that the Cauchy distributions and delta measures are the only fixed points of the homomorphisms.

A multiplicative free convolution \boxtimes and Boolean convolution \boxplus and their associated infinite divisibility, which we explain more in Section 2, can be defined on the positive real line and on the unit circle. So, in relation to multiplicative free infinite divisibility, a natural question is if a divisibility indicator and a composition semigroup exist for these convolutions.

The main subject of the second part of this paper is the existence of a counterpart of the semigroup of [9] for the multiplicative convolutions. Remarkably, the same commutation relation used in the additive case is true for multiplicative convolutions. In contrast to the additive convolutions, some difficulties appear for multiplicative convolutions on the positive real line and on the unit circle. On the unit circle, the problem is the non-uniqueness of convolution powers: neither $\mu^{\boxtimes t}$ nor $\mu^{\boxplus t}$ can be defined uniquely [7, 20]. On the positive real line, a difficulty comes from the fact that $\mu^{\boxplus t}$ cannot be defined for large t [10]. We, however, manage to define composition semigroups and free divisibility indicators for multiplicative convolutions, following the additive case.

Moreover, we prove several results analogous to the additive case. We define free divisibility indicators for these convolutions. For instance, a probability measure on the unit circle is freely infinitely divisible concerning the multiplicative free convolution if and only if the indicator is not less than one. We also prove how the multiplicative divisibility indicator changes under free and Boolean powers and then the multiplicative analogue of Bożejko's conjecture. We also prove that the Poisson kernel on the unit circle, which corresponds to the Cauchy distribution on the real line, has the multiplicative free divisibility indicator equal to infinity.

This paper is organized as follows. In Section 2 we explain on additive and multiplicative convolutions, both free and Boolean. Section 3 is devoted to the study of the additive free divisibility indicator. In Section 4 we develop the multiplicative counterparts of composition semigroups. Combining results of [9] and Section 4, we have simple commutation relations between various pairs of convolutions: the additive free convolution and the additive Boolean one; the multiplicative free convolution and the multiplicative Boolean one. In Section 5, we prove more commutation relations between additive convolutions and multiplicative convolutions. These properties provide new examples of freely infinitely divisible distributions with respect to both multiplicative and additive free convolutions. In the Appendix, the Cauchy distributions and point measures are shown to be the only fixed points of the Boolean-to-free Bercovici-Pata bijection.

2 Preliminaries

2.1 Additive free convolution

Let $\mathcal{P}(\mathbb{R})$ denote the set of the probability measures on \mathbb{R} . The upper half-plane and the lower half-plane are respectively denoted as \mathbb{C}_+ and \mathbb{C}_- . In this article, $G_\mu(z)$ denotes the Cauchy transform $\int_{\mathbb{R}} \frac{\mu(dx)}{z-x}$ of a probability measure μ and $F_\mu(z)$ denotes its reciprocal $\frac{1}{G_\mu(z)}$. An additive free convolution was introduced by Voiculescu in [30] for compactly supported measures and later extended to all probability measures in [13]. Let $\phi_\mu(z)$ be the Voiculescu transform of μ defined by

$$\phi_\mu(z) = F_\mu^{-1}(z) - z$$

for z in a suitable open set of \mathbb{C}_+ . The free convolution $\mu \boxplus \nu$ of probability measures μ and ν is characterized by $\phi_{\mu \boxplus \nu}(z) = \phi_\mu(z) + \phi_\nu(z)$. It is known that a probability measure μ is freely (\boxplus - for short) infinitely divisible if and only if ϕ_μ has an analytic continuation to \mathbb{C}_+ with values in $\mathbb{C}_- \cup \mathbb{R}$. The free infinite divisibility of μ is also equivalent to the existence of probability measures $\mu^{\boxplus t}$ for $0 \leq t < \infty$ which satisfy $\phi_{\mu^{\boxplus t}}(z) = t\phi_\mu(z)$.

If μ is \boxplus -infinitely divisible, then it has the Lévy-Khintchine representation

$$\phi_\mu(z) = \gamma_\mu + \int_{\mathbb{R}} \frac{1+xz}{z-x} \tau_\mu(dx),$$

where $\gamma_\mu \in \mathbb{R}$ and τ_μ is a non-negative finite measure. An infinitely divisible distribution can also be characterized by the limit of an infinitesimal triangular array [19] as in probability theory.

2.2 Multiplicative free convolutions

Let \mathbb{R}_+ , \mathbb{D} and \mathbb{T} denote the positive real line $[0, \infty)$, the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and the unit circle of the complex plane, respectively. Moreover, let $\mathcal{P}(\mathbb{R}_+)$ and $\mathcal{P}(\mathbb{T})$ denote the sets of probability measures on \mathbb{R}_+ and \mathbb{T} , respectively. For probability measures μ, ν on \mathbb{T} and \mathbb{R}_+ , multiplicative free convolutions $\mu \boxtimes \nu$ on \mathbb{T} and \mathbb{R}_+ were respectively introduced in [31] for compactly supported probability measures. $\mu \boxtimes \nu$ on \mathbb{T} is the distribution of UV , where U and V are unitary free random variables with distributions μ and ν , respectively. Similarly, $\mu \boxtimes \nu$ on \mathbb{R}_+ is defined as the distribution of $X^{1/2}YX^{1/2}$, where X and Y are positive, self-adjoint free random variables with distributions μ and ν , respectively.

The multiplicative convolution of probability measures on \mathbb{R}_+ with non-compact supports was considered in [13].

A probability measure μ on \mathbb{T} (resp. \mathbb{R}_+) is said to be \boxtimes -infinite divisible if for any $n \in \mathbb{N}$, there is μ_n on \mathbb{T} (resp. \mathbb{R}_+) such that $\mu = \mu_n^{\boxtimes n} = \mu_n \boxtimes \cdots \boxtimes \mu_n$.

To investigate multiplicative convolutions, an important transform is

$$\eta_\mu(z) = \frac{\psi_\mu(z)}{1 + \psi_\mu(z)}$$

for $\mu \in \mathcal{P}(\mathbb{R}_+)$ or $\mathcal{P}(\mathbb{T})$, where $\psi_\mu(z)$ is a moment generating function defined by $\int_{\text{supp } \mu} \frac{tz}{1-tz} \mu(dt)$. The transform η_μ is characterized as follows [7].

Proposition 2.1. (i) Let $\eta : \mathbb{C} \setminus \mathbb{R}_+ \rightarrow \mathbb{C}$ be an analytic map satisfying $\eta(\bar{z}) = \overline{\eta(z)}$ and $\eta \neq 0$. Then the following properties are equivalent.

(1) $\eta = \eta_\mu$ for a probability measure $\mu \in \mathcal{P}(\mathbb{R}_+)$, $\mu \neq \delta_0$.

(2) $\eta(-0) = 0$ and $\arg \eta(z) \in [\arg z, \pi)$ for any $z \in \mathbb{C}_+$.

(ii) Let $\eta : \mathbb{D} \rightarrow \mathbb{C}$ be an analytic map. Then the following properties are equivalent.

(1) $\eta = \eta_\mu$ for a probability measure $\mu \in \mathcal{P}(\mathbb{T})$.

(2) $|\eta(z)| \leq |z|$ for $z \in \mathbb{D}$.

We can define $\eta_\mu^{-1}(z)$ in a suitable open set and then define

$$\Sigma_\mu(z) := \frac{\eta_\mu^{-1}(z)}{z}.$$

Multiplicative free convolutions \boxtimes are characterized by the multiplications of the functions Σ : $\Sigma_{\mu \boxtimes \nu} = \Sigma_\mu \Sigma_\nu$. μ on \mathbb{T} is \boxtimes -infinitely divisible if and only if Σ_μ can be written as [12]

$$\Sigma_\mu(z) = \gamma_\mu \exp \left(\int_{\mathbb{T}} \frac{1 + \zeta z}{1 - \zeta z} \tau_\mu(d\zeta) \right)$$

for $z \in \mathbb{D}$, where $\gamma_\mu \in \mathbb{T}$ and τ_μ is a non-negative finite measure.

On the positive real line, μ is \boxtimes -infinitely divisible if and only if Σ_μ can be written as [13]

$$\Sigma_\mu(z) = \exp \left(-a_\mu z + b_\mu + \int_{\mathbb{R}_+} \frac{1+xz}{z-x} \tau_\mu(dx) \right), \quad (2.1)$$

where $a_\mu \geq 0$, $b_\mu \in \mathbb{R}$ and τ_μ is a non-negative finite measure on $\mathbb{R}_+ = [0, \infty)$.

In both cases of \mathbb{R}_+ and \mathbb{T} , \boxtimes -infinite divisibility is also equivalent to the existence of a weakly continuous convolution semigroup $\mu^{\boxtimes t}$ for $t \geq 0$ with $\mu^{\boxtimes 0} = \delta_1$. Another characterization of infinite divisible distributions is found in [14] in terms of infinitesimal triangular arrays.

2.3 Additive and multiplicative Boolean convolutions

Additive and multiplicative Boolean convolutions on \mathbb{T} were introduced in [28] and [20] respectively. Let $K_\mu(z)$ be the energy function [28] defined by

$$K_\mu(z) = z - F_\mu(z)$$

for $\mu \in \mathcal{P}(\mathbb{R})$. The Boolean convolution $\mu \uplus \nu$ is characterized by $K_{\mu \uplus \nu}(z) = K_\mu(z) + K_\nu(z)$. A Boolean power $\mu^{\uplus t}$ can be defined for any $t > 0$ and any probability measure μ . The Lévy-Khintchine representation is written as [28]

$$K_\mu(z) = \gamma_\mu + \int_{\mathbb{R}} \frac{1+xz}{z-x} \tau_\mu(dx),$$

where γ_μ and τ_μ satisfy the same conditions as the free case. We note that η_μ and K_μ are related through the formula $\eta_\mu(z) = zK_\mu(\frac{1}{z})$.

Let $k_\mu(z) := \frac{z}{\eta_\mu(z)}$. The multiplicative Boolean convolution $\mu \boxtimes \nu$ is characterized by

$$k_{\mu \boxtimes \nu} = k_\mu k_\nu.$$

A probability measure $\mu \in \mathcal{P}(\mathbb{T})$ is said to be \boxtimes -infinitely divisible if for any $n \in \mathbb{N}$, we can find μ_n such that $\mu = \mu_n^{\boxtimes n}$. This is equivalent to the condition that $\frac{1}{k_\mu(z)}$ does not have a zero point of \mathbb{D} , and also equivalent to the existence of the Lévy-Khintchine formula [20]

$$k_\mu(z) = \gamma_\mu \exp \left(\int_{\mathbb{T}} \frac{1 + \zeta z}{1 - \zeta z} \tau_\mu(d\zeta) \right), \quad (2.2)$$

where γ_μ and τ_μ satisfy the same conditions as the free case.

Results on infinitesimal triangular arrays can be found in [32].

Bercovici proved in [10] that the multiplicative Boolean convolution does not preserve $\mathcal{P}(\mathbb{R}_+)$. However, there still exists a Boolean power $\mu^{\boxtimes t} \in \mathcal{P}(\mathbb{R}_+)$ for $0 \leq t \leq 1$.

3 On the free divisibility indicator

We focus on a composition semigroup $\{\mathbb{B}_t\}_{t \geq 0}$, introduced by Belinschi and Nica [9], defined to be

$$\mathbb{B}_t(\mu) = \left(\mu^{\boxplus(1+t)} \right)^{\boxtimes \frac{1}{1+t}}.$$

In addition to the semigroup property, \mathbb{B}_t is a homomorphism regarding the multiplicative free convolution \boxtimes : $\mathbb{B}_t(\mu \boxtimes \nu) = \mathbb{B}_t(\mu) \boxtimes \mathbb{B}_t(\nu)$ for probability measures μ, ν , one of which is supported on \mathbb{R}_+ . It is known, moreover, that \mathbb{B}_1 coincides with the Bercovici-Pata bijection Λ_B from the Boolean convolution to the free one. The reader is referred to [11] for the definition of Λ_B . Let $\phi(\mu)$ denote the free divisibility indicator defined by

$$\phi(\mu) := \sup\{t \geq 0 : \mu \in \mathbb{B}_t(\mathcal{P}(\mathbb{R}))\}.$$

For a probability measure μ , Belinschi and Nica proved that a probability measure ν uniquely exists such that $\mathbb{B}_{\phi(\mu)}(\nu) = \mu$. Therefore, $\mathbb{B}_t(\mu)$ can be defined as a probability measure for any $t \geq -\phi(\mu)$. The free divisibility indicator satisfies the following properties [9].

Theorem 3.1. (1) $\mu^{\boxplus t}$ exists for $t \geq \max\{1 - \phi(\mu), 0\}$.
(2) μ is \boxplus -infinitely divisible if and only if $\phi(\mu) \geq 1$.
(3) $\phi(\mathbb{B}_t(\mu))$ can be calculated as

$$\phi(\mathbb{B}_t(\mu)) = \phi(\mu) + t \quad (3.1)$$

for $t \geq -\phi(\mu)$.

The following property was crucial to prove the semigroup property of \mathbb{B}_t [9]. This will be also crucial in Theorem 3.3 below.

Proposition 3.2. *Let $\mu \in \mathcal{P}(\mathbb{R})$ and p, q be two real numbers such that $p \geq 1$ and $1 - \frac{1}{p} < q$. Then*

$$(\mu^{\boxplus p})^{\boxplus q} = (\mu^{\boxplus q'})^{\boxplus p'}, \quad (3.2)$$

where p', q' are defined by $p' := pq/(1 - p + pq)$, $q' := 1 - p + pq$.

3.1 On free powers, Boolean powers and shifts

We can explicitly calculate the free divisibility indicator for free and Boolean time evolutions.

Theorem 3.3. *Let μ be a probability measure of $\mathcal{P}(\mathbb{R})$. Then $\phi(\mu^{\boxplus t}) = \frac{1}{t}\phi(\mu)$ for $t > 0$. Moreover, $\phi(\mu^{\boxplus t}) - 1 = \frac{1}{t}(\phi(\mu) - 1)$ for $t > \max\{1 - \phi(\mu), 0\}$.*

Proof. Suppose $\phi(\mu) = t$, then there is ν such that $\mathbb{B}_t(\nu) = \mu$ and then

$$\begin{aligned} \mu^{\boxplus s} &= \left((\nu^{\boxplus (1+t)})^{\boxplus \frac{s+t}{1+t}} \right)^{\boxplus \frac{s}{s+t}} \\ &= \left((\nu^{\boxplus s})^{\boxplus \frac{s+t}{s}} \right)^{\boxplus \frac{s}{s+t}} \\ &= \left((\nu^{\boxplus s})^{\boxplus (1+t/s)} \right)^{\boxplus \frac{1}{1+t/s}} \\ &= \mathbb{B}_{t/s}(\nu^{\boxplus s}), \end{aligned}$$

where Proposition 3.2 was applied in the second line with $p = 1 + t$, $q = \frac{s+t}{1+t}$. Therefore, $\phi(\mu^{\boxplus s}) \geq t/s = \frac{\phi(\mu)}{s}$. Replacing s by $1/s$ and μ by $\mu^{\boxplus s}$, we see that $\phi(\mu) \geq s\phi(\mu^{\boxplus s})$. Therefore, the conclusion follows for Boolean powers. From this result and (3.1), we can prove that $\phi(\mu^{\boxplus (t+1)}) = \phi(\mathbb{B}_t(\mu)^{\boxplus (1+t)}) = \frac{1}{1+t}\phi(\mathbb{B}_t(\mu)) = \frac{\phi(\mu)+t}{1+t}$ for $t > \max\{-\phi(\mu), -1\}$. \square

The first identity of Theorem 3.3 leads to a new interpretation of $\phi(\mu)$ in terms of Boolean powers. Let us mention that this characterization is very useful for deriving the value of the free divisibility indicator, as we will see in the next section. This is because, in practice, it is much easier to calculate Boolean powers than free powers.

Corollary 3.4. $\phi(\mu) = \sup\{t \geq 0 : \mu^{\boxplus t} \text{ is } \boxplus\text{-infinitely divisible}\}$.

Bożejko's conjecture follows immediately.

Proposition 3.5. *If μ is \boxplus -infinitely divisible, then so is $\mu^{\boxplus t}$ for $0 \leq t \leq 1$. Moreover,*

$$\phi_{\mu^{\boxplus t}}(z) = K_{(\mu^{\boxplus (1-t)})^{\boxplus t/(1-t)}}(z) = \frac{t}{1-t} K_{\mu^{\boxplus (1-t)}}(z)$$

for $0 < t < 1$. In terms of the Boolean Bercovici-Pata bijection Λ_B ,

$$\Lambda_B\left((\mu^{\boxplus (1-t)})^{\boxplus t/(1-t)}\right) = \mu^{\boxplus t}.$$

Proof. Infinite divisibility is immediate from Theorem 3.3. For $t \in (0, 1)$,

$$\begin{aligned}\Lambda_B\left((\mu^{\boxplus(1-t)})^{\boxplus t/(1-t)}\right) &= \mathbb{B}_1((\mu^{\boxplus(1-t)})^{\boxplus t/(1-t)}) \\ &= \left(\left((\mu^{\boxplus(1-t)})^{\boxplus t/(1-t)}\right)^{\boxplus 2}\right)^{\boxplus 1/2} \\ &= \left(\left((\mu^{\boxplus 2t})^{\boxplus 1/2}\right)^{\boxplus 2}\right)^{\boxplus 1/2} \\ &= \mu^{\boxplus t},\end{aligned}$$

where Proposition 3.2 was applied in the third line. \square

Free divisibility indicators are invariant under shifts of probability measures and also under the Boolean convolutions with delta measures. Let us first prove the following.

Lemma 3.6. *Let μ be a probability measure on \mathbb{R} . Then the following are equivalent:*

- (1) μ is freely infinitely divisible;
- (2) $\mu \boxplus \delta_a$ is freely infinitely divisible for any $a \in \mathbb{R}$.

Proof. It suffices to prove that (1) implies (2). From (1), there exists a ν such that $\mu = \Lambda_B(\nu)$. Since $\Lambda_B = \mathbb{B}_1$ is the Bercovici-Pata bijection from Boolean to free,

$$\phi_\mu(z) = F_\mu^{-1}(z) - z = z - F_\nu(z) \quad (3.3)$$

by definition. We note here that $\mu \boxplus \delta_a$ is characterized by

$$F_{\mu \boxplus \delta_a}(z) = F_\mu(z) - a. \quad (3.4)$$

If there exists a λ such that $\Lambda_B(\lambda) = \mu \boxplus \delta_a$, λ should be characterized by $F_{\Lambda_B(\lambda)}(z) = F_\mu(z) - a$, so that by $F_{\Lambda_B(\lambda)}^{-1}(z) = F_\mu^{-1}(z + a)$. From (3.3) and (3.4), this is also equivalent to

$$F_\lambda(z) = -2a + F_\nu(z + a).$$

Since $F_\nu(z + a) = F_{\nu \boxplus \delta_{-a}}(z)$, such a λ indeed exists, defined by $\lambda = (\nu \boxplus \delta_{-a}) \boxplus \delta_{2a}$. This implies the free infinite divisibility of $\mu \boxplus \delta_a$. \square

Corollary 3.4, combined with the above lemma, enables us to prove the invariance of $\phi(\mu)$ under shifts and the Boolean convolution with δ_a .

Proposition 3.7. *Let μ be a probability measure on \mathbb{R} and $a \in \mathbb{R}$. Then $\phi(\mu) = \phi(\mu \boxplus \delta_a) = \phi(\mu \boxplus \delta_a)$.*

Proof. For $t \geq 0$, we have $(\mu \boxplus \delta_a)^{\boxplus t} = \mu^{\boxplus t} \boxplus \delta_{at}$. Lemma 3.6 then implies that $\mu^{\boxplus t}$ is freely infinitely divisible if and only if $(\mu \boxplus \delta_a)^{\boxplus t}$ is so. Therefore, $\phi(\mu) = \phi(\mu \boxplus \delta_a)$ from Corollary 3.4.

For the second identity, we prove that

$$(\mu \boxplus \delta_a)^{\boxplus t} = (\mu^{\boxplus t} \boxplus \delta_a) \boxplus \delta_{(t-1)a}.$$

Indeed,

$$\begin{aligned}
F_{(\mu \boxplus \delta_a)^{\boxplus t}}(z) &= (1-t)z + tF_\mu(z-a) \\
&= (1-t)(z-a) + tF_\mu(z-a) - (t-1)a \\
&= F_{\mu^{\boxplus t}}(z-a) - (t-1)a.
\end{aligned}$$

Again from Lemma 3.6, $(\mu \boxplus \delta_a)^{\boxplus t}$ is freely infinitely divisible if and only if $\mu^{\boxplus t} \boxplus \delta_a$ is so. The latter assertion is obviously equivalent to the free infinite divisibility of $\mu^{\boxplus t}$. Therefore, $\phi(\mu) = \phi(\mu \boxplus \delta_a)$ from Corollary 3.4. \square

3.2 On Jacobi parameters and free Meixner laws

In this section, we give an upper bound for the free divisibility indicator in terms of Jacobi parameters. We start from the definition of Jacobi parameters [22]. For a probability measure μ with all finite moments, let us orthogonalize the sequence $(1, x, x^2, x^3, \dots)$ in the Hilbert space $L^2(\mathbb{R}, \mu)$, following the Gram-Schmidt method. This procedure yields orthogonal polynomials $(P_0(x), P_1(x), P_2(x), \dots)$ with $\deg P_n(x) = n$. Multiplying constants, we take P_n to be monic, i.e., the coefficient of x^n is one. It is known that they satisfy a recurrence relation

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_{n-1} P_{n-1}(x)$$

for $n \geq 0$, under the convention that $P_{-1}(x) = 0$. The coefficients β_n and γ_n are called Jacobi parameters and they satisfy $\beta_n \in \mathbb{R}$ and $\gamma_n \geq 0$. Indeed, it is known that $\gamma_0 \cdots \gamma_n = \int_{\mathbb{R}} |P_{n+1}(x)|^2 \mu(dx)$ for $n \geq 0$. Moreover, $N := |\text{supp } \mu| < \infty$ if and only if $\gamma_{N-1} = 0$ and $\gamma_n > 0$ for $m = 0, \dots, N-2$. We write Jacobi parameters as

$$J(\mu) = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 & \dots \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \dots \end{pmatrix}.$$

Continued fraction representation of G_μ can be expressed in terms of the Jacobi Parameters:

$$\int_{\mathbb{R}} \frac{\mu(dx)}{z-x} = \frac{1}{z - \beta_0 - \frac{\gamma_0}{z - \beta_1 - \frac{\gamma_1}{z - \beta_2 - \dots}}}.$$

This is useful to calculate G_μ from the Jacobi parameters.

In this section, we also consider Jacobi parameters $\beta_0, \beta_1, \gamma_0, \gamma_1$ for a probability measure μ with a finite fourth moment. We can define them just by orthogonalizing the sequence $(1, x, x^2)$ in $L^2(\mathbb{R}, \mu)$. The resulting orthogonal monic polynomials $(P_0(x), P_1(x), P_2(x))$ are expressed as

$$P_0(x) = 1, \quad P_1(x) = x - \beta_0, \quad P_2(x) = (x - \beta_0)(x - \beta_1) - \gamma_0$$

for some real numbers $\beta_0, \beta_1, \gamma_0$. Then $\gamma_0 = \int_{\mathbb{R}} |P_1(x)|^2 \mu(dx)$. β_0 is the mean of μ and γ_0 coincides with the variance of μ . If $\gamma_0 = 0$, we define $\gamma_1 := 0$, and if $\gamma_0 \neq 0$, we define $\gamma_1 := \gamma_0^{-1} \int_{\mathbb{R}} |P_2(x)|^2 \mu(dx)$.

Proposition 3.8. *Let μ be a probability measure with a finite fourth moment. Let us consider the Jacobi parameters $(\gamma_i, \beta_i)_{i=0,1}$ of μ . Then $\phi(\mu) \leq \gamma_1/\gamma_0$.*

Proof. It was observed in [18] that the Boolean power by t is nothing else than multiplying both β_0 and γ_0 by t . On the other hand, it was proved in [25] that if a probability measure is \boxplus -infinitely divisible, then $\gamma_0 \leq \gamma_1$ (this is still true under the assumption of finite fourth moment). Therefore, $\phi(\mu) \leq \gamma_1/\gamma_0$ from Theorem 3.3. \square

The last proposition extends Theorem 3.7 in [5] which gives an upper bound for the divisibility indicator in terms of the Boolean kurtosis: $Kurt^\boxplus(\mu) \geq \phi(\mu)$. Indeed, for a probability measure with mean zero and finite fourth moment, we have $Kurt^\boxplus(\mu) = (\gamma_1 + \beta_1^2)/\gamma_0 \geq \gamma_1/\gamma_0$. Therefore, Proposition 3.8 gives a sharper estimate.

Example 3.9. The family of q -Gaussian distributions introduced by Bożejko and Speicher in [17] (see also the paper [16] of Bożejko, Kümmerer and Speicher) interpolate between the normal ($q = 1$) and the semicircle ($q = 0$) laws. It is determined in terms of their orthogonal polynomials $H_{n,q}(x)$, called the q -Hermite polynomials, via the recurrence relation

$$xH_{n,q}(x) = H_{n+1,q}(x) + \frac{1 - q^n}{1 - q}H_{n-1,q}(x).$$

It was proved in [2] that the q -Gaussian distributions are freely infinitely divisible for all $q \in [0, 1]$. A direct application of the Proposition 3.8 gives the estimate $\phi(\mu) \leq \gamma_1/\gamma_0 = 1 + q$. In particular, this shows that the q -Gaussian distributions are not freely infinitely divisible when $q \in [-1, 0)$.

Notice from last example that the semicircle distribution satisfies the equality $\phi(\mu) = \gamma_1/\gamma_0$. In the next example we will see that the class of free Meixner distributions also satisfy this property.

Example 3.10. The free Meixner distributions $\mu_{\beta_0, \gamma_0, \beta_1, \gamma_1}$ are probability measures with Jacobi parameters

$$J(\mu) = \begin{pmatrix} \beta_0 & \beta_1 & \beta_1 & \beta_1 & \dots \\ \gamma_0 & \gamma_1 & \gamma_1 & \gamma_1 & \dots \end{pmatrix}$$

where $\beta_0, \beta_1 \in \mathbb{R}$ and $\gamma_0, \gamma_1 \geq 0$. More explicitly, $\mu_{0,1,b,1+c}$ is written as

$$\frac{1}{2\pi} \cdot \frac{\sqrt{[4(1+c) - (x-b)^2]_+}}{1 + bx + cx^2} dx + (0, 1 \text{ or } 2 \text{ atoms}),$$

where $f(x)_+$ is defined by $\max\{f(x), 0\}$. General free Meixner distributions are affine transformations of this case; see [1], [15].

Let $\gamma = \gamma_1 - \gamma_0$ and $\beta = \beta_1 - \beta_0$. Then

$$J(\mu^{\boxplus t}) = \begin{pmatrix} \beta_1 t & \beta + \beta_0 t & \beta + \beta_0 t & \beta + \beta_0 t & \dots \\ \gamma_0 t & \gamma + \gamma_0 t & \gamma + \gamma_0 t & \gamma + \gamma_0 t & \dots \end{pmatrix}, \quad (3.5)$$

as shown in [3]. On the other hand, from [18], we have

$$J(\mu^{\boxplus s}) = \begin{pmatrix} \beta_0 s, & \beta_1, & \beta_1, & \beta_1, & \dots \\ \gamma_0 s, & \gamma_1, & \gamma_1, & \gamma_1, & \dots \end{pmatrix}. \quad (3.6)$$

Now, it is easy to derive the free divisibility indicator of $\mu_{\beta_0, \gamma_0, \beta_1, \gamma_1}$. Let us denote by $\gamma_i(\nu)$ Jacobi parameters of ν to distinguish a probability measure. We get $\gamma_1(\mu_{\beta_0, \gamma_0, \beta_1, \gamma_1}^{\boxplus t}) - \gamma_0(\mu_{\beta_0, \gamma_0, \beta_1, \gamma_1}^{\boxplus t}) = \gamma_1(\mu_{\beta_0, \gamma_0, \beta_1, \gamma_1}) - t\gamma_0(\mu_{\beta_0, \gamma_0, \beta_1, \gamma_1})$ using (3.6). By the way, Saitoh and Yoshida [27] showed that a free Meixner distribution μ is \boxplus -infinitely divisible if and only if $\gamma_1(\mu) - \gamma_0(\mu) \geq 0$. Therefore, we conclude that

$$\phi(\mu_{\beta_0, \gamma_0, \beta_1, \gamma_1}) = \gamma_1(\mu_{\beta_0, \gamma_0, \beta_1, \gamma_1}) / \gamma_0(\mu_{\beta_0, \gamma_0, \beta_1, \gamma_1})$$

from Corollary 3.4. In particular, for a so-called Kesten-McKay distribution μ_t , with absolutely continuous part

$$\frac{1}{2\pi} \cdot \frac{\sqrt{4t - x^2}}{1 - (1 - t)x^2},$$

we get $\phi(\mu_t) = t$. This distribution was studied by Kesten [23] in connection to simple random walks on free groups. They appear in free probability theory as free additive powers of a Bernoulli distribution. For relations between d -regular graphs and these distributions, see [24].

As shown in the above example, every free Meixner distribution achieves the upper bound of Proposition 3.8. More strongly, this characterizes free Meixner distributions.

Theorem 3.11. *Let μ be a probability measure with a finite fourth moment and Jacobi parameters $(\gamma_i, \beta_i)_{i=0,1}$. Then $\phi(\mu) = \gamma_1/\gamma_0$ if and only if μ is a free Meixner distribution.*

Proof. Let us use the notation $\gamma_i(\mu)$ to distinguish probability measures. We assume that $\phi(\mu) = \gamma_1(\mu)/\gamma_0(\mu)$. If $\gamma_0(\mu) = 0$, μ is a delta measure at a point, so that $\phi(\mu) = \infty$. We assume that the variance $\gamma_0(\mu)$ is nonzero.

First, let us suppose that $\gamma_1(\mu) = 0$. From the paragraph previous to Proposition 3.8, the L^2 -norm of $P_2(x)$ is zero, which implies that μ is supported on at most two points. In other words, μ is of the form $p\delta_a + (1 - p)\delta_b$, i.e., μ is a Bernoulli law. This is a free Meixner law since its Jacobi parameters are given by

$$J(\mu) = \begin{pmatrix} \beta_0(\mu), & \beta_1(\mu), & \beta_1(\mu), & \beta_1(\mu), & \dots \\ \gamma_0(\mu), & 0, & 0, & 0, & \dots \end{pmatrix},$$

where $\beta_0(\mu) = pa + (1 - p)b$, $\gamma_0(\mu) = p(1 - p)(b - a)^2$ and $\beta_1(\mu) = pb + (1 - p)a$. This calculation can be checked by using the continued fraction of the Stieltjes transform.

Next let us assume that $t := \phi(\mu) = \gamma_1(\mu)/\gamma_0(\mu) > 0$. Then there exists ν such that $\mathbb{B}_t(\nu) = \mu$. Using the relation $\phi(\mathbb{B}_t(\mu)) = \phi(\mu) + t$, we conclude $\phi(\nu) = 0$. Since γ_0 is equal to variance, $\gamma_0(\mu) = \gamma_0(\nu^{\boxplus(1+t)})/(1 + t) = \gamma_0(\nu)$. From the relation (3.14) of [25] and (3.9) of [18],

$$\begin{aligned} \gamma_1(\mu) &= \gamma_1(\nu^{\boxplus(1+t)}) \\ &= \gamma_1(\nu) + t\gamma_0(\nu). \end{aligned}$$

Divided by $\gamma_0(\mu) = \gamma_0(\nu)$, the above equality becomes

$$\frac{\gamma_1(\mu)}{\gamma_0(\mu)} = \frac{\gamma_1(\nu)}{\gamma_0(\nu)} + t.$$

Using the assumption $\phi(\mu) = \gamma_1(\mu)/\gamma_0(\mu)$, we have

$$\phi(\mu) = t = \frac{\gamma_1(\mu)}{\gamma_0(\mu)} = \frac{\gamma_1(\nu)}{\gamma_0(\nu)} + t.$$

Therefore, $\gamma_1(\nu) = 0$, so that ν is a Bernoulli law. (3.5) and (3.6) imply that the set of the free Meixner laws is closed under the operation \mathbb{B}_t , so that $\mu = \mathbb{B}_t(\nu)$ is also a free Meixner law. \square

3.3 On free stable laws and Boolean stable laws

We investigate free and Boolean stable distributions in terms of free divisibility indicators. Free and Boolean stable laws were respectively introduced in [13] and [28], and their domains of attraction were studied in [11]. Using Theorem 3.3, we show that the free divisibility indicator of a Boolean strictly stable distribution takes only two possible values, 0 or ∞ . Similarly, in the free case we get two possibilities, 1 or ∞ .

Let D_t denote the dilation operation defined by $(D_t\mu)(B) := \mu(t^{-1}B)$ for every Borel set B . Since, $\mathbb{B}_t(D_s(\mu)) = D_s(\mathbb{B}_t(\mu))$, for $t, s > 0$, $\phi(\mu)$ does not change under dilations.

In this paper, a probability measure ν_α is said to be \uplus -stable of index α if $\nu_\alpha \uplus \nu_\alpha = D_{1/2^\alpha}(\nu_\alpha) \uplus \delta_b$, for some $b \in \mathbb{R}$. If $b = 0$ the probability measure ν_α is said to be \uplus -strictly stable of index α . Now it is clear from Theorem 3.3 and Proposition 3.7 that ν_α satisfies

$$\frac{1}{2}\phi(\nu_\alpha) = \phi(\nu_\alpha \uplus \nu_\alpha) = \phi(D_{1/2^\alpha}(\nu_\alpha)) = \phi(\nu_\alpha)$$

which is possible only if $\phi(\nu_\alpha) = 0$ or $\phi(\nu_\alpha) = \infty$, depending on whether μ is freely infinitely divisible or not. Note that both 0 and ∞ can be achieved since the Cauchy distribution ($\alpha = 1$) is freely infinitely divisible and the Bernoulli distribution ($\alpha = 2$) is not.

Analogously, a probability measure σ_α is said to be \boxplus -stable of index α if $\sigma_\alpha \boxplus \sigma_\alpha = D_{1/2^\alpha}(\sigma_\alpha) \boxplus \delta_b$, for some $b \in \mathbb{R}$. Also, σ_α is said to be \boxplus -strictly stable when $b = 0$. In exactly the same manner, as for the Boolean case, we see that

$$\frac{1}{2}(1 - \phi(\sigma_\alpha)) = (1 - \phi(\sigma_\alpha)).$$

This yields only two possibilities, either $\phi(\sigma_\alpha) = 1$ or $\phi(\sigma_\alpha) = \infty$. Also in the free case, both values 1 and ∞ can be achieved: a semicircle law ($\alpha = 2$) satisfy $\phi(\sigma_2) = 1$ [9]; a Cauchy distribution σ_1 , which is \boxplus -strictly stable of index $\alpha = 1$, satisfy $\phi(\sigma_1) = \infty$. Later we also consider free and Boolean 1/2-stable laws.

Remark 3.12. Until now, we have not been able to say much about classical stable laws, except for the Cauchy distributions. However, an upper bound can be estimated for the free divisibility indicator of the Gaussian distribution $N(0, 1)$, i.e., the 2-stable law in probability theory. Indeed, one can show, using Theorem 3.3 and numerical computations of cumulants¹, that $\phi(N(0, 1)) < 1.2$. The lower bound $\phi(N(0, 1)) \geq 1$, which is much harder to prove, is implicit in Belinschi et al. [8].

The free divisibility indicators of Cauchy and delta distributions are infinity since they are the fixed points of \mathbb{B}_t [9]. The following theorem shows that there are other measures with their free divisibility indicators equal to ∞ , while they are not fixed points of \mathbb{B}_t .

Theorem 3.13. *Any \oplus -stable distribution ν of index $1/2$ is \boxplus -infinitely divisible. Moreover, $\phi(\nu) = \infty$.*

Proof. Thanks to Proposition 3.7, we may assume that ν is \oplus -strictly stable. The reciprocal Cauchy transform of ν is given by [28]

$$F_\nu(z) = z + bz^{1/2}, \quad 0 \leq \arg b \leq \frac{\pi}{2}.$$

The Voiculescu transform then becomes

$$\phi_\nu(z) = \frac{b^2}{2} - \sqrt{b^2 z + b^4/4}.$$

$\phi_\nu(z)$ can be extended to a continuous function from $(\mathbb{C}_+ \cup \mathbb{R}) \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$. In addition, this mapping is homeomorphic. To understand the situation, it helps us to define $\psi := J \circ \phi_\nu \circ J^{-1}$ as a function in $\overline{\mathbb{D}} = \{|z| \leq 1\}$, where $J : \mathbb{C}_+ \rightarrow \mathbb{D}$ is an analytic isomorphism. To prove ν is freely infinitely divisible, it is sufficient to prove that $\text{Im } \phi_\nu(x)$ takes non-positive values on \mathbb{R} . Indeed, let us suppose $\text{Im } \phi_\nu(x) \leq 0$ for $x \in \mathbb{R}$. This assumption is equivalent to $\psi(\partial\mathbb{D}) \subset \mathbb{D}^c = \{z \in \mathbb{C} : |z| \geq 1\}$. From the homeomorphic property, $\psi(\partial\mathbb{D})$ is equal to $\partial\psi(\mathbb{D})$, and moreover, is a Jordan curve. Therefore, $\psi(\partial\mathbb{D})$ divides \mathbb{C} into two connected, simply connected open sets. One does not intersect with $\overline{\mathbb{D}}$ and the other includes \mathbb{D} . We can easily observe that $\lim_{y \rightarrow \infty} \text{Im } \phi_\nu(iy) = -\infty$, which implies that $\psi(\mathbb{D})$ coincides with the first one.

Now, we prove that $\text{Im } \phi_\nu(z) \leq 0$ for any $x \in \mathbb{R}$. Let $c = b^2$ and suppose, without loss of generality, that $c = e^{i\theta}$ ($0 \leq \theta \leq \pi$). Then

$$\text{Im } \phi_\nu(x) = \frac{\sin \theta}{2} - \frac{1}{\sqrt{2}} \left[\left(x^2 + \frac{\cos \theta}{2} x + \frac{1}{16} \right)^{1/2} - x \cos \theta - \frac{\cos 2\theta}{4} \right]^{1/2}.$$

Let us define

$$f(x) := \left(x^2 + \frac{\cos \theta}{2} x + \frac{1}{16} \right)^{1/2} - x \cos \theta - \frac{\cos 2\theta}{4}.$$

¹The authors would like to thank Professor Franz Lehner for his assistance to improve these numerical computations.

It is easy to prove that f decreases in $(-\infty, 0)$ and increases in $(0, \infty)$ with its minimum value $f(0) = \frac{\sin^2 \theta}{2}$. Therefore, $\operatorname{Im} \phi_\nu(x)$ takes the maximum value $\operatorname{Im} \phi_\nu(0) = 0$, and hence $\operatorname{Im} \phi_\nu(x) \leq 0$ for any $x \in \mathbb{R}$ as we wanted. The fact $\phi(\nu) = \infty$ now follows from the discussion prior to Remark 3.12. \square

Corollary 3.14. *Any free $1/2$ -stable distribution σ satisfies $\phi(\sigma) = \infty$.*

Proof. A \boxplus -stable law σ of index $1/2$ is just $\Lambda_B(\nu)$ where ν is a \boxplus -stable law of index $1/2$. Therefore, we see that $\phi(\sigma) = \phi(\nu) + 1 = \infty + 1 = \infty$. \square

4 Composition semigroups for multiplicative convolutions

In this section, we prove that many results on additive convolutions have counterparts for multiplicative convolutions. A transformation $f_\mu(z) = \log(\eta_\mu(e^z))$ is useful to understand such results intuitively. Indeed, in terms of f_μ , multiplicative convolutions can be characterized in a way analogous to the additive ones. For instance, the multiplicative free and Boolean convolutions can be characterized by

$$f_{\mu \boxtimes \nu}^{-1}(z) = f_\mu^{-1}(z) + f_\nu^{-1}(z) - z, \quad f_{\mu \boxtimes \nu}(z) = f_\mu(z) + f_\nu(z) - z.$$

Therefore, $f_\mu(z)$, $f_\mu^{-1}(z) - z$ and $f_\mu(z) - z$ play the same roles as the reciprocal Cauchy transform, the Voiculescu transform and the energy function, respectively. From this observation, we can expect many results on multiplicative convolutions.

However, we cannot avoid the following problems.

- (A) On the unit circle, $\mu^{\boxtimes t}$ for $t > 0$ and $\mu^{\boxtimes t}$ for $t > 1$ can be defined for any \boxtimes -infinitely divisible measure μ , but these powers are not unique [7, 20].
- (B) On the positive real line, Boolean powers of a generic probability measure can be defined only for a finite time [10].

We discuss the above problems in this section.

The essence of the problem (A) can be understood in terms of the universal covering of the Riemannian surface $\mathbb{D} \setminus \{0\}$. We, however, do not use such concepts to avoid introducing many terminologies.

4.1 A composition semigroup on the unit circle

Let $\mathcal{ID}(\boxtimes; \mathbb{T})$ be the set of \boxtimes -infinitely divisible distributions on \mathbb{T} . Results in this subsection are sometimes trivial for the normalized Haar measure ω , so that we define the set $\mathcal{ID}(\boxtimes; \mathbb{T})_0 := \mathcal{ID}(\boxtimes; \mathbb{T}) \setminus \{\omega\}$. This set is important for multiplicative free powers as well as for Boolean ones. On the unit circle \mathbb{T} , the multiplicative Bercovici-Pata bijection

from \boxtimes to \boxdot was considered in a paper [32]. We denote that map by Λ_{MB} and then Λ_{MB} satisfies

$$k_\mu(z) = \Sigma_{\Lambda_{MB}(\mu)}(z). \quad (4.1)$$

The bijection Λ_{MB} is a homeomorphism from $\mathcal{ID}(\boxtimes; \mathbb{T})$ to the set of the \boxdot -infinitely divisible distributions.

For $\mu \in \mathcal{ID}(\boxtimes; \mathbb{T})_0$, both $\mu^{\boxtimes t}$ for $t > 1$ and $\mu^{\boxtimes t}$ for $t > 0$ can be defined, but they are not unique. For the Boolean case, this ambiguity is due to the rotational freedom [20]. Because of this ambiguity, Boolean powers do not work well in some situations [21]. However, we can overcome this difficulty, introducing a family of countably many free and Boolean powers. Let $u_\mu^{(0)}$ be the function satisfying $k_\mu(z) = e^{u_\mu^{(0)}(z)}$ with $-\pi < \text{Im } u_\mu^{(0)}(0) < \pi$. If one needs to consider $\text{Im } u_\mu^{(0)}(0) = -\pi$ or π , one can approximate $u_\mu^{(0)}$ using a sequence of probability measures μ_n satisfying $\text{Im } u_{\mu_n}^{(0)}(0) \searrow -\pi$ or $\text{Im } u_{\mu_n}^{(0)}(0) \nearrow \pi$. We can define a family of convolution semigroups $\mu^{\boxtimes n t}$ ($n \in \mathbb{Z}$) from the relation

$$k_{\mu^{\boxtimes n t}}(z) = e^{t u_\mu^{(n)}(z)},$$

where $u_\mu^{(n)} = -2\pi n i + u_\mu^{(0)}$.

Also in the free case, the ambiguity comes from the rotational freedom. We define free powers in terms of subordination functions. Let $\Phi_t^{(n)}(z) := z e^{(t-1)u_\mu^{(n)}(z)}$ for $\mu \in \mathcal{ID}(\boxtimes; \mathbb{T})_0$ and $t > 1$. An analytic map $\omega_t^{(n)} : \mathbb{D} \rightarrow \mathbb{D}$ exists for any $t > 1$ satisfying the following properties (see Theorem 3.5 of [7]):

$$(\Omega 1) \quad \Phi_t^{(n)}(\omega_t^{(n)}(z)) = z \text{ for } z \in \mathbb{D},$$

$$(\Omega 2) \quad |\omega_t^{(n)}(z)| \leq |z| \text{ in } \mathbb{D}.$$

$$(\Omega 3) \quad \omega_t^{(n)} \text{ is univalent.}$$

The condition $(\Omega 1)$ implies the uniqueness of $\omega_t^{(n)}$. Then a free power $\mu^{\boxtimes n t}$ ($n \in \mathbb{Z}$) can be defined by the relation

$$\eta_{\mu^{\boxtimes n t}}(z) = \eta_\mu(\omega_t^{(n)}(z)).$$

Because of this formula, we call $\omega_t^{(n)}$ an *n*th subordination function. From the property $(\Omega 3)$, $\eta_{\mu^{\boxtimes n t}}$ does not vanish in $\mathbb{D} \setminus \{0\}$. Therefore, $\mu^{\boxtimes n t}$ also belongs to $\mathcal{ID}(\boxtimes; \mathbb{T})_0$. If μ is the normalized Haar measure ω , then a convolution power is unique and is just the ω itself, so that we do not need to distinguish the countably many powers.

We define an analogue of the semigroup \mathbb{B}_t , paying attention to the rotational freedom. For a real number φ , we define $[\varphi]$ to be an integer determined as follows:

- (1) $[\varphi] = 0$ if $-\pi < \varphi < \pi$;
- (2) $[\varphi] = n$ if $n\pi \leq \varphi < (n+2)\pi$ and $n \geq 1$;
- (3) $[\varphi] = n$ if $n\pi \geq \varphi > (n-2)\pi$ and $n \leq -1$.

In particular, $[-\varphi] = -[\varphi]$. Using this, we define continuous families of Boolean and free powers:

$$\mu^{\boxtimes \varphi t} := \mu^{\boxtimes [\varphi] t}, \quad \mu^{\boxdot \varphi t} := \mu^{\boxdot [\varphi] t}$$

for $\varphi \in \mathbb{R}$.

Definition 4.1. A family of maps $\{\mathbb{M}_t^{(n)}\}_{t \geq 0}$ from $\mathcal{ID}(\mathbb{U}; \mathbb{T})$ into itself is defined by

$$\mathbb{M}_t^{(n)}(\mu) = (\mu^{\boxtimes_{\arg m_1(\mu)}(t+1)})^{\boxtimes_{(t+1) \arg m_1(\mu)} \frac{1}{t+1}}, \quad (4.2)$$

where $m_1(\mu) := \int_{\mathbb{T}} \zeta \mu(d\zeta)$ and $\arg m_1(\mu)$ is taken to satisfy $n = [\arg m_1(\mu)]$. If $\arg m_1(\mu) = n\pi$, we can define $\mathbb{M}_t^{(n)}(\mu)$ to be the limit $\lim_{r \searrow 0} \mathbb{M}_t^{(n)}(\mu \boxtimes \delta_{e^{i r \text{sign}(n)}})$.

Remark 4.2. The essence of the above definition is to make the function $m_1(\mathbb{M}_t^{(n)}(\mu))$ constant with respect to $t \in \mathbb{R}_+$. For this purpose, we allow the argument of the first moment to take any number of \mathbb{R} , not only of $(-\pi, \pi)$. This idea is also important in Proposition 4.3.

$\mathbb{M}_t^{(n)}$ is injective on $\mathcal{ID}(\mathbb{U}; \mathbb{T})$ for $t \geq 0$, but not continuous on $\mathcal{ID}(\mathbb{U}; \mathbb{T})$ for $t \notin \mathbb{N}$. This discontinuity has the same origin as the function z^t . In this paper we take the principal values, so that $\mathbb{M}_t^{(n)}$ is continuous in the subset $\{\mu \in \mathcal{ID}(\mathbb{U}; \mathbb{T}) : m_1(\mu) \in \overline{\mathbb{D}} \setminus [-1, 0]\}$. $\mathbb{M}_1^{(n)}$ coincides with the Bercovici-Pata bijection Λ_{MB} for any n , as in the additive case. Therefore, $\mathbb{M}_k^{(n)}$ is also continuous on $\mathcal{ID}(\mathbb{U}; \mathbb{T})$ for any $k \in \mathbb{N}$ and does not depend on n , since $\mathbb{M}_k^{(n)}$ is the iteration of Λ_{MB} by k times (see Theorem 4.4).

The following is a key to the semigroup property of $\mathbb{M}_t^{(n)}$ for each $n \in \mathbb{Z}$.

Proposition 4.3. Let $\mu \in \mathcal{ID}(\mathbb{U}; \mathbb{T})_0$ and $\arg m_1(\mu) \in \mathbb{R}$ be an arbitrary argument of $m_1(\mu)$.

(1) Let p, q be two real numbers such that $p \geq 1$ and $1 - \frac{1}{p} < q$. Then we have

$$(\mu^{\boxtimes_{\arg m_1(\mu)} p})^{\boxtimes_{p \arg m_1(\mu)} q} = (\mu^{\boxtimes_{\arg m_1(\mu)} q'})^{\boxtimes_{q' \arg m_1(\mu)} p'}, \quad (4.3)$$

where p', q' are defined by $p' := pq/(1 - p + pq)$, $q' := 1 - p + pq$.

(2) $(\mu^{\boxtimes_{\arg m_1(\mu)} t})^{\boxtimes_{t \arg m_1(\mu)} s} = \mu^{\boxtimes_{\arg m_1(\mu)} ts}$ for $t, s \geq 1$.

(3) $(\mu^{\boxtimes_{\arg m_1(\mu)} t})^{\boxtimes_{t \arg m_1(\mu)} s} = \mu^{\boxtimes_{\arg m_1(\mu)} ts}$ for $t, s \geq 0$.

Proof. We use the notation $z_{\arg z}^t$ to distinguish the branches. More precisely, $z_{\arg z}^t$ is defined to be $e^{it \arg z + t \log |z|}$ for any argument of $z \neq 0$. We note that

$$(zw)_{\arg z + \arg w}^t = z_{\arg z}^t w_{\arg w}^t \quad (4.4)$$

for any $z, w \neq 0$.

We first prove the following fact: for the $[\arg m_1(\mu)]$ th subordination function ω_t associated to a probability measure $\mu \in \mathcal{ID}(\mathbb{U}; \mathbb{T})_0$, one gets

$$\frac{\omega_t(z)}{z} = \left(\frac{\eta_{\mu^{\boxtimes_{\arg m_1(\mu)} t}}(z)}{z} \right)^{1-1/t}, \quad \frac{\eta_{\mu^{\boxtimes_{\arg m_1(\mu)} t}}(z)}{z} = \left(\frac{\omega_t(z)}{z} \right)^{\frac{t}{t-1}}_{(t-1) \arg m_1(\mu)}. \quad (4.5)$$

This is proved as follows. For simplicity, let $n := [\arg m_1(\mu)]$. From the relation $\Phi_t^{(n)} \circ \omega_t^{(n)}(z) = z$ and (4.4), we have

$$\begin{aligned} 1 &= \frac{\omega_t^{(n)}(z)}{z} \left(\frac{\omega_t^{(n)}(z)}{z} \frac{z}{\eta_{\mu^{\boxtimes n} t}(z)} \right)^{t-1} \\ &= \frac{\omega_t^{(n)}(z)}{z} \left(\frac{\omega_t^{(n)}(z)}{z} \right)^{(t-1) \arg m_1(\mu)} \left(\frac{z}{\eta_{\mu^{\boxtimes n} t}(z)} \right)^{t-1} \\ &= \left(\frac{\omega_t^{(n)}(z)}{z} \right)^t \left(\frac{z}{\eta_{\mu^{\boxtimes n} t}(z)} \right)^{t-1} \end{aligned}$$

Therefore, the desired relations follow.

For simplicity, let us introduce the notations $\lambda := (\mu^{\boxtimes \arg m_1(\mu) p})^{\boxtimes p \arg m_1(\mu) q}$ and $\nu := (\mu^{\boxtimes \arg m_1(\mu) q'})^{\boxtimes q' \arg m_1(\mu) p'}$. If the $\arg m_1(\mu)$ th subordination function for μ is denoted simply by ω_t , one obtains

$$\frac{\eta_\lambda(z)}{z} = \left(\frac{\eta_{\mu^{\boxtimes \arg m_1(\mu) p}}(z)}{z} \right)^q_{p \arg m_1(\mu)} = \left(\frac{\omega_p(z)}{z} \right)^{\frac{pq}{p-1}}_{(p-1) \arg m_1(\mu)},$$

where (4.5) was applied. On the other hand, if the $[q' \arg m_1(\mu)]$ th subordination function for $\rho := \mu^{\boxtimes \arg m_1(\mu) q'}$ is denoted by σ_t , then (Ω1) implies that $\frac{\eta_\rho(\sigma_{p'}(z))}{z} = \left(\frac{\sigma_{p'}(z)}{z} \right)^{\frac{1}{p'-1}}_{q'(p'-1) \arg m_1(\mu)}$.

We note that $q'(p'-1) = p-1$, and therefore,

$$\begin{aligned} \frac{\eta_\nu(z)}{z} &= \frac{\eta_\rho(\sigma_{p'}(z))}{z} = \frac{\eta_\rho(\sigma_{p'}(z))}{\sigma_{p'}(z)} \frac{\sigma_{p'}(z)}{z} \\ &= \left(\frac{\sigma_{p'}(z)}{z} \right)^{\frac{p'}{p'-1}}_{(p-1) \arg m_1(\mu)} = \left(\frac{\sigma_{p'}(z)}{z} \right)^{\frac{pq}{p-1}}_{(p-1) \arg m_1(\mu)}. \end{aligned}$$

The above calculations have reduced the problem to proving $\omega_p = \sigma_{p'}$. Let us prove this. The second identity of (4.5), μ and ω_t replaced by $\rho = \mu^{\boxtimes \arg m_1(\mu) q'}$ and σ_t respectively, leads to

$$\frac{\sigma_{p'}(z)}{z} = \left(\frac{\eta_\mu(\sigma_{p'}(z))}{\sigma_{p'}(z)} \right)^{p-1}_{\arg m_1(\mu)}.$$

This relation says that $\sigma_{p'}$ is exactly the right inverse of $\Phi_p(z) := z \left(\frac{z}{\eta_\mu(z)} \right)^{p-1}_{\arg m_1(\mu)}$. Therefore, $\sigma_{p'} = \omega_p$ from the uniqueness.

(2) and (3) follow easily from analogous and simpler arguments. \square

The semigroup property of $\mathbb{M}_t^{(n)}$ is immediate from Proposition 4.3.

Theorem 4.4. $\mathbb{M}_{t+s}^{(n)} = \mathbb{M}_t^{(n)} \circ \mathbb{M}_s^{(n)}$ on $\mathcal{ID}(\boxtimes; \mathbb{T})$, $t, s \geq 0$, $n \in \mathbb{Z}$.

Proof. Let us assume that $n = \lceil \arg m_1(\mu) \rceil$. Let us take $p = \frac{t+s+1}{t+1}$ and $q = \frac{s+1}{t+s+1}$ in Proposition 4.3 (1) and replace μ by $\mu^{\boxtimes_{\arg m_1(\mu)}(t+1)}$. Then Proposition 4.3 (1) and (2) say that

$$\begin{aligned} & \left((\mu^{\boxtimes_{\arg m_1(\mu)}(t+1)})^{\boxtimes_{(t+1)\arg m_1(\mu)} \frac{1}{t+1}} \right)^{\boxtimes_{\arg m_1(\mu)}(s+1)} \\ &= \left((\mu^{\boxtimes_{\arg m_1(\mu)}(t+1)})^{\boxtimes_{(t+1)\arg m_1(\mu)} \frac{t+s+1}{t+1}} \right)^{\boxtimes_{(t+s+1)\arg m_1(\mu)} \frac{s+1}{t+s+1}} \\ &= (\mu^{\boxtimes_{\arg m_1(\mu)}(t+s+1)})^{\boxtimes_{(t+s+1)\arg m_1(\mu)} \frac{s+1}{t+s+1}}. \end{aligned} \quad (4.6)$$

Therefore $\mathbb{M}_s^{(n)} \circ \mathbb{M}_t^{(n)}$ can be calculated as

$$\begin{aligned} \mathbb{M}_s^{(n)} \circ \mathbb{M}_t^{(n)}(\mu) &= \left(\left((\mu^{\boxtimes_{\arg m_1(\mu)}(t+1)})^{\boxtimes_{(t+1)\arg m_1(\mu)} \frac{1}{t+1}} \right)^{\boxtimes_{\arg m_1(\mu)}(s+1)} \right)^{\boxtimes_{(s+1)\arg m_1(\mu)} \frac{1}{s+1}} \\ &= (\mu^{\boxtimes_{\arg m_1(\mu)}(t+s+1)})^{\boxtimes_{(t+s+1)\arg m_1(\mu)} \frac{1}{t+s+1}} \\ &= \mathbb{M}_{t+s}^{(n)}(\mu), \end{aligned}$$

where Proposition 4.3 (3) and (4.6) were applied in the second line. \square

An analogue of the free additive divisibility indicator can be defined for the multiplicative case as follows.

Definition 4.5. We define a multiplicative free divisibility indicator $\theta^{(n)}(\mu)$ to be

$$\theta^{(n)}(\mu) = \sup\{t \geq 0 : \mu \in \mathbb{M}_t^{(n)}(\mathcal{ID}(\boxtimes; \mathbb{T}))\}$$

for $\mu \in \mathcal{ID}(\boxtimes; \mathbb{T})$.

$\mathbb{M}_t^{(n)}(\mu)$ is well defined for $t \geq -\theta^{(n)}(\mu)$ from the same reason as the additive case.

We prove counterparts of Theorems 3.1 and 3.3.

Theorem 4.6. We consider a probability measure $\mu \in \mathcal{ID}(\boxtimes; \mathbb{T})$.

- (1) $\mu^{\boxtimes_n t}$ exists for $t \geq \max\{1 - \theta^{(n)}(\mu), 0\}$.
- (2) μ is \boxtimes -infinitely divisible if and only if $\theta^{(0)}(\mu) \geq 1$.
- (3) $\theta^{(n)}(\mathbb{M}_t^{(n)}(\mu)) = \theta^{(n)}(\mu) + t$ for $t \geq -\theta^{(n)}(\mu)$.
- (4) $\theta^{(\lceil t \arg m_1(\mu) \rceil)}(\mu^{\boxtimes_{\arg m_1(\mu)} t}) = \frac{1}{t} \theta^{(\lceil \arg m_1(\mu) \rceil)}(\mu)$ for $t > 0$ and any $\arg m_1(\mu) \in \mathbb{R}$.
- (5) $\theta^{(\lceil t \arg m_1(\mu) \rceil)}(\mu^{\boxtimes_{\arg m_1(\mu)} t}) - 1 = \frac{1}{t} (\theta^{(\lceil \arg m_1(\mu) \rceil)}(\mu) - 1)$ for $t > \max\{1 - \theta^{(\lceil \arg m_1(\mu) \rceil)}(\mu), 0\}$ and any $\arg m_1(\mu) \in \mathbb{R}$.

Remark 4.7. Since $M_k^{(n)}$ does not depend on n for any positive integer k , the statement (2) is also equivalent to $\theta^{(n)}(\mu) \geq 1$ for any fixed n .

Proof. All the proofs are similar to the additive case with slight modification. The reader is referred to Section 5 of [9] and Theorem 3.3 of this paper. For instance, (4) can be proved as follows. Let us take $\arg m_1(\mu)$ such that $n = \lceil \arg m_1(\mu) \rceil$ and suppose $\theta^{(n)}(\mu) = t$. By

definition, there exists ν such that $\mathbb{M}_t^{(n)}(\nu) = \mu$. We note that $\arg m_1(\mu) = \arg m_1(\nu)$. Then

$$\begin{aligned} \mu^{\boxtimes_{\arg m_1(\mu)} s} &= \left((\nu^{\boxtimes_{\arg m_1(\mu)} (1+t)})^{\boxtimes_{(t+1) \arg m_1(\mu)} \frac{s+t}{1+t}} \right)^{\boxtimes_{(t+s) \arg m_1(\mu)} \frac{s}{s+t}} \\ &= \left((\nu^{\boxtimes_{\arg m_1(\mu)} s})^{\boxtimes_{s \arg m_1(\mu)} \frac{s+t}{s}} \right)^{\boxtimes_{(t+s) \arg m_1(\mu)} \frac{s}{s+t}} \\ &= \left((\nu^{\boxtimes_{\arg m_1(\mu)} s})^{\boxtimes_{s \arg m_1(\mu)} (1+t/s)} \right)^{\boxtimes_{(t+s) \arg m_1(\mu)} \frac{1}{1+t/s}} \\ &= \mathbb{M}_{t/s}^{([s \arg m_1(\mu)])} (\nu^{\boxtimes_{\arg m_1(\mu)} s}), \end{aligned}$$

where we used Proposition 4.3 (1) with $p = 1+t$, $q = \frac{s+t}{1+t}$. Therefore, $\theta^{([s \arg m_1(\mu)])}(\mu^{\boxtimes_n s}) \geq t/s = \frac{\theta^{(n)}(\mu)}{s}$. As in Theorem 3.1, let us replace s by $1/s$ and μ by $\mu^{\boxtimes_{\arg m_1(\mu)} s}$. Using Proposition 4.3(2), we obtain the converse inequality. \square

We can immediately characterize the free divisibility indicator with Boolean multiplicative powers on the unit circle.

Corollary 4.8. $\theta^{(n)}(\mu) = \sup\{t \geq 0 : \mu^{\boxtimes_n t} \text{ is } \boxtimes\text{-infinitely divisible}\}$ for $\mu \in \mathcal{ID}(\boxtimes; \mathbb{T})$.

An analogue of Bożejko's conjecture for multiplicative convolutions is also the case on the unit circle.

Proposition 4.9. If $\mu \in \mathcal{ID}(\boxtimes; \mathbb{T})$ is \boxtimes -infinitely divisible, then so is $\mu^{\boxtimes_n t}$ for $0 \leq t \leq 1$ and $n \in \mathbb{Z}$. Moreover, $\mu^{\boxtimes_n t} = \Lambda_{MB}((\mu^{\boxtimes_n (1-t)})^{\boxtimes_{(1-t) \arg m_1(\mu)} t/(1-t)})$ for $0 < t < 1$, where $n = [\arg m_1(\mu)]$.

Proof. The proof is similar to Proposition 3.5. \square

Example 4.10. For $a \geq 0$ and $b \in \mathbb{R}$, let μ be a probability measure on \mathbb{T} defined by

$$\mu(d\theta) = \frac{1}{2\pi} \frac{1 - e^{-2a}}{1 + e^{-2a} - 2e^{-a} \cos(\theta - b)} d\theta, \quad 0 \leq \theta < 2\pi.$$

This is an analogue of the Cauchy distribution on \mathbb{R} since μ is identical to the density of the Poisson kernel. Let $c := e^{-a+ib}$, then $\eta_\mu(z) = cz$ and

$$\Sigma_\mu(z) = k_\mu(z) = c^{-1} = e^{-ib} \exp \left(a \int_{\mathbb{T}} \frac{1 + \zeta z}{1 - \zeta z} \omega(d\zeta) \right),$$

where $\omega(d\theta)$ is the normalized Haar measure. Therefore, $\mathbb{M}_t^{(n)}(\mu) = \mu$ for any $n \in \mathbb{Z}$ and any $t \geq 0$. The free divisibility indicator $\theta^{(n)}(\mu)$ is equal to ∞ .

4.2 A composition semigroup on the positive real line

From Proposition 2.1(i), a logarithm $\log k_\mu(z) = \log(z/\eta_\mu(z))$ can be defined in $\mathbb{C} \setminus \mathbb{R}_+$ with values in \mathbb{C} for $\mu \in \mathcal{P}(\mathbb{R}_+)$. The function $\log k_\mu(z)$ maps \mathbb{C}_+ to $\mathbb{C}_- \cup \mathbb{R}$, and therefore, it has the Pick-Nevanlinna representation

$$\log k_\mu(z) = -a_\mu z + b_\mu + \int_0^\infty \frac{1+xz}{z-x} \tau_\mu(dx) \quad (4.7)$$

for $a_\mu \geq 0$, $b_\mu \in \mathbb{R}$ and a non-negative finite measure τ_μ on \mathbb{R}_+ . This is, in a sense, a Lévy-Khintchine formula for the multiplicative Boolean convolution on \mathbb{R}_+ . To understand a Bercovici-Pata bijection, we have to know when a function $K(z) = -az + b + \int_0^\infty \frac{1+xz}{z-x} \tau(dx)$ can be written as $\log k_\mu(z)$ for a probability measure μ on \mathbb{R}_+ . For instance, Proposition 2.1 implies that $\text{Im}(\log k_\mu(z)) \in (-\pi + \arg z, 0]$ in \mathbb{C}_+ . In particular, $-\pi \leq \text{Im}(\log k_\mu) \leq 0$. Therefore, $a = 0$. Moreover, τ cannot contain the singular part: if the singular part were non-zero, a point $x_0 \geq 0$ would exist such that $\text{Im} K(x_0 + i0) = \infty$. These conditions, however, are far from a complete characterization.

In spite of the above, we can still construct an injective mapping Λ_{MB} from $\mathcal{P}(\mathbb{R}_+)$ to the set of \boxtimes -infinitely divisible distributions with the relation

$$k_\mu(z) = \Sigma_{\Lambda_{MB}(\mu)}(z). \quad (4.8)$$

Let us call this map Λ_{MB} a Bercovici-Pata map from \mathfrak{M} to \boxtimes for probability measures on the positive real line. As explained above, $a_{\Lambda_{MB}(\mu)} = 0$ and $\tau_{\Lambda_{MB}(\mu)}$ is absolutely continuous with respect to the Lebesgue measure, where a_ν and τ_ν have been defined in (2.1). Therefore, Λ_{MB} is not surjective.

Now we define an analogue of the semigroup \mathbb{B}_t for the multiplicative convolutions.

Definition 4.11. A family of maps $\{\mathbb{M}_t\}_{t \geq 0}$ from $\mathcal{P}(\mathbb{R}_+)$ into itself is defined by

$$\mathbb{M}_t(\mu) = (\mu^{\boxtimes(t+1)})^{\boxtimes \frac{1}{t+1}}. \quad (4.9)$$

As proved in [10], $\mu^{\boxtimes t} \in \mathcal{P}(\mathbb{R})$ is defined for any probability measure $\mu \in \mathcal{P}(\mathbb{R}_+)$ and $0 \leq t \leq 1$. Moreover, it is easy to prove that $\mu^{\boxtimes t}$ in fact belongs to $\mathcal{P}(\mathbb{R}_+)$. Therefore, \mathbb{M}_t is well defined. The map \mathbb{M}_t on the positive real line is simpler than on the unit circle, since a Boolean power is unique if exists.

The following result is essentially the same as the additive case, except for the restriction $q \leq 1$.

Proposition 4.12. Let p, q be two real numbers such that $p \geq 1$ and $1 - \frac{1}{p} < q \leq 1$. We have

$$(\mu^{\boxtimes p})^{\boxtimes q} = (\mu^{\boxtimes q'})^{\boxtimes p'}, \quad (4.10)$$

where $\mu \in \mathcal{P}(\mathbb{R}_+)$ and p', q' are defined by $p' := pq/(1 - p + pq)$, $q' := 1 - p + pq$. All the convolution powers are well defined under the above assumptions.

The proof is easier than that of Proposition 4.3; we do not have to pay attention to branches of analytic mappings. The semigroup property holds also in this case.

Theorem 4.13. $\mathbb{M}_{t+s} = \mathbb{M}_t \circ \mathbb{M}_s$ on $\mathcal{P}(\mathbb{R}_+)$, $t, s \geq 0$.

We can also define a free divisibility indicator:

$$\theta(\mu) = \sup\{t \geq 0 : \mu \in \mathbb{M}_t(\mathcal{P}(\mathbb{R}_+))\}$$

for $\mu \in \mathcal{P}(\mathbb{R}_+)$. Some results of Theorem 4.6 have no counterparts for probability measures on the positive real line. This is because the Bercovici-Pata map is not surjective and a Boolean power cannot be defined for a large time. We can, however, still prove the following.

Proposition 4.14. *Let μ be a probability measure on $\mathcal{P}(\mathbb{R}_+)$. Then*

- (1) $\mu^{\boxplus t}$ exists for $t \geq \max\{1 - \theta(\mu), 0\}$,
- (2) μ is \boxtimes -infinitely divisible if $\theta(\mu) \geq 1$,
- (3) $\theta(\mathbb{M}_t(\mu)) = \theta(\mu) + t$ for $t \geq -\theta(\mu)$.

Proof. All the proofs are similar to the previous cases (see Theorem 4.6 of this paper and Section 5 of [9]). A remark on (2) is that the \boxtimes -infinite divisibility of $\mu \in \mathcal{P}(\mathbb{R}_+)$ does not imply $\theta(\mu) \geq 1$ since the Bercovici-Pata map is not surjective. \square

5 Commutation relations between Boolean and free convolution powers

We have seen that convolution powers for \boxtimes and \boxplus on \mathbb{R}_+ satisfy the same commutation relation as the additive case (see Propositions 3.2, 4.12). Moreover, there are other commutation relations involving free and Boolean powers. We will prove such relations in this section. These are useful to construct new examples of \boxtimes -infinitely divisible distributions.

Proposition 5.1. *The following commutation relations hold for $\mu \in \mathcal{P}(\mathbb{R}_+)$.*

- (1) $(\mu^{\boxplus t})^{\boxtimes s} = D_{t^{s-1}}(\mu^{\boxtimes s})^{\boxplus t}$ for $t \geq 1$ and $s \geq 1$.
- (2) $(\mu^{\boxplus t})^{\boxtimes s} = D_{t^{s-1}}(\mu^{\boxtimes s})^{\boxplus t}$ for $t \geq 0$ and $s \geq 1$.
- (3) $(\mu^{\boxplus t})^{\boxtimes s} = (\mu^{\boxtimes s})^{\boxplus t^s}$ for $t \geq 0$ and $s \leq 1$.

Proof. We note that $\mu^{\boxplus t}$ is supported on \mathbb{R}_+ for $t \geq 1$ (see discussions in Subsection 2.4 of [9]). (1) and (2) were essentially proved in Proposition 3.5 of [9].

(3) We recall the relations $\eta_{\mu^{\boxplus t}}(z) = t\eta_{\mu}(z)$ and $\eta_{\mu^{\boxtimes s}}(z)/z = (\eta_{\mu}(z)/z)^s$. For the left hand side, we have

$$\frac{\eta_{(\mu^{\boxplus t})^{\boxtimes s}}(z)}{z} = \left(\frac{\eta_{\mu^{\boxplus t}}(z)}{z} \right)^s = t^s \left(\frac{\eta_{\mu}(z)}{z} \right)^s,$$

and for the right hand side,

$$\frac{\eta_{(\mu^{\boxtimes s})^{\boxplus t^s}}(z)}{z} = t^s \frac{\eta_{\mu^{\boxtimes s}}(z)}{z} = t^s \left(\frac{\eta_{\mu}(z)}{z} \right)^s.$$

Therefore, they coincide. \square

Remark 5.2. (i) The parameter t in (1) may not be extended to $t \geq 0$ even if μ is \boxplus -infinitely divisible. This is because $\text{supp } \mu \subset [0, \infty)$ does not imply $\text{supp } \mu^{\boxplus t} \subset [0, \infty)$ for every $t \geq 0$. We will discuss this problem in another paper [4].

(ii) In Propositions 3.2, 4.12 and 5.1, we have derived five commutation relations among \boxplus , \boxtimes , \uplus and \boxdot . The only missing relation is for the pair \boxplus and \boxdot . The question of if there is an algebraic relation between these two convolutions is an open problem.

The following result is immediate.

Theorem 5.3. *If $\mu \in \mathcal{P}(\mathbb{R}_+)$ is \boxtimes -infinitely divisible, then so are $\mu^{\boxplus t}$ for $t \geq 1$ and $\mu^{\uplus s}$ for $s \geq 0$.*

It is well known that the free Poisson distribution π , characterized by $\phi_\pi(z) = \frac{z}{z-1}$, is both \boxplus and \boxtimes -infinitely divisible. The following example generalizes this fact for different powers of π .

Example 5.4. Let $\pi_{t,s,r} := ((\pi^{\boxplus t})^{\boxtimes s})^{\uplus r}$ for $r, s, t \geq 0$. It is clear from Theorem 5.3 that $\pi_{t,s,r}$ is \boxtimes -infinitely divisible for $r, s \geq 0$ and $t \geq 1$. Moreover, since $\pi^{\boxplus t}$ is supported on \mathbb{R}_+ for every $t > 0$, combining Propositions 3.5 and 5.1, we see that $\pi_{t,s,r}$ is \boxplus -infinitely divisible for $r \leq 1$, $s \geq 1$ and $t \geq 0$. In particular, if $r \leq 1$ and $s, t \geq 1$, $\pi_{t,s,r}$ is infinitely divisible with respect to both \boxplus and \boxtimes .

On the other hand, $\pi^{\boxplus t}$ is not \boxtimes -infinitely divisible for $t < 1$ as shown by Pérez-Abreu and Sakuma; see Proposition 10 of [26]. This shows that we cannot extend Theorem 5.3 to $t < 1$, even if $\mu^{\boxplus t}$ exists.

Appendix: Fixed points of \mathbb{B}_t

As we have shown, some measures have free divisibility indicators infinity as well as Cauchy distributions. So one may ask if this is because of some fixed point property. In this section we determine all the fixed points of the Boolean-to-free Bercovici-Pata bijection and more generally of \mathbb{B}_t for each $t > 0$. The key is the following lemma regarding analytic maps from \mathbb{C}_+ to \mathbb{C}_+ .

Lemma 5.5. *Let $F : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ be an analytic function with $\text{Im } F(z) > \text{Im } z$ such that*

$$F(z) - z = F(F(z)) - F(z), \quad z \in \mathbb{C}_+. \quad (5.1)$$

Then $F(z) = z + a + ib$, for some $a \in \mathbb{R}$ and $b > 0$.

Proof. (1) F is injective. Indeed if $F(z) = F(w)$ then $F(F(z)) = F(F(w))$ and then from Equation (5.1) $z = 2F(z) - F(F(z)) = 2F(w) - F(F(w)) = w$.

(2) Let $z_0 \in \mathbb{C}_+$ and suppose that $F(z_0) - z_0 = a + bi$. Moreover, suppose that F is not identically equal to $z + a + bi$. Then $G(z) := F(z) - z$ is analytic and not constant. Let D be a bounded domain such that $\overline{D} \subset \mathbb{C}_+$ then, by the Identity Theorem, $G(z) = a + bi$ for at most a finite number of points. Hence, there exists a radius r , such that the ball $B_r(z_0)$

satisfies that $F(z) - z \neq a + bi$ for all $z \in \overline{B_r(z_0)} \setminus \{z_0\}$. Moreover we may assume that $F(z_0) \notin \overline{B_r(z_0)}$ since $F(z_0) \neq z_0$ by the inequality $\operatorname{Im} F(z) > \operatorname{Im} z$. Since F is injective, $F^{on+1}(z_0) \notin F^n(\overline{B_r(z_0)})$ for any $n \geq 1$. Let us consider the curve $C = \partial B_r(z_0)$. Since C is compact, there exist $t > 0$ such that $|f(z) - z - a + bi| > t$ for all $z \in C$.

Take an arbitrary $z \in C$. If we write $F(z) - z = c + di$, then $|(c + id) - (a + ib)| > t$. From the iterative use of (5.1), we have $F^{on}(z) = z + n(c + id)$ and $F^{on}(z_0) = z_0 + n(a + ib)$ and then we see that $|F^{on}(z) - F^{on}(z_0)| > tn - |z_0| - |z|$ which is as large as we wish. However, for each $n > 0$, $F^n(C)$ is a simple curve surrounding $F^n(z_0)$ and $F^{n+1}(z_0) - F^n(z_0) = a + bi$ is bounded. So, for n large enough, $F^n(C)$ must surround $F^{n+1}(z_0)$, contradicting the fact $F^{on+1}(z_0) \notin F^n(\overline{B_r(z_0)})$ \square

Theorem 5.6. *Let $t > 0$ be real and μ be a fixed point of \mathbb{B}_t , i.e. $\mathbb{B}_t(\mu) = \mu$ then μ is a point measure or a Cauchy distribution $\gamma_{a,b}$ with density*

$$\gamma_{a,b}(x)dx = \frac{b}{\pi[(x-a)^2 + b^2]}dx, \quad x \in \mathbb{R}$$

for some $a \in \mathbb{R}$, $b > 0$.

Proof. Suppose μ is not a point measure, then $\operatorname{Im} F(z) > \operatorname{Im} z$ for any $z \in \mathbb{C}_+$. From the basic properties of free and boolean convolutions,

$$F_{\mu^{\boxplus(t+1)}}^{-1}(z) = (t+1)F_{\mu}^{-1}(z) - tz \quad (5.2)$$

and

$$F_{\mu^{\boxplus(t+1)}}(z) = (t+1)F_{\mu}(z) - tz. \quad (5.3)$$

Recall that $\mathbb{B}_t(\mu) = \mu$ is equivalent to $\mu^{\boxplus(t+1)} = \mu^{\boxplus(t+1)}$. Plugging (5.3) into (5.2) we have

$$z = F_{\mu^{\boxplus(t+1)}}^{-1}(F_{\mu^{\boxplus(t+1)}}(z)) = (t+1)F_{\mu}^{-1}((t+1)F_{\mu}(z) - tz) - t((t+1)F_{\mu}(z) - tz),$$

from which

$$F_{\mu}^{-1}((t+1)F_{\mu}(z) - tz) = tF_{\mu}(z) - (t-1)z.$$

Applying F_{μ} to both sides of the previous equation we get

$$(t+1)F_{\mu}(z) - tz = F_{\mu}(tF_{\mu}(z) - (t-1)z)$$

or

$$F_{\mu}(z) - z = F_{\mu}(tF_{\mu}(z) - (t-1)z) - tF_{\mu}(z) + (t-1)z.$$

Now, let $W_{\mu}(z) = F_{\mu^{\boxplus t}}(z) = tF_{\mu}(z) - (t-1)z$ then

$$F_{\mu}(z) - z = F_{\mu}(W_{\mu}(z)) - W_{\mu}(z).$$

Multiplied by t , this equation becomes

$$W_{\mu}(z) - z = W_{\mu}(W_{\mu}(z)) - W_{\mu}(z).$$

This equation is exactly Equation (5.1) for $F = W_{\mu}$ which satisfies the assumptions of Lemma 5.5. So, $F_{\mu^{\boxplus t}}$ is of the form $z + a_0 + ib_0$ for some $a_0 \in \mathbb{R}$ and $b_0 > 0$. This, in turn, implies that $F_{\mu}(z) = z + a + ib$, where $a = \frac{a_0}{t}$ and $b = \frac{b_0}{t}$, i.e. $\mu = \gamma_{a,b}$. \square

Corollary 5.7. *The Boolean-to-free Bercovici-Pata bijection Λ_B has no periodic points of order greater than one.*

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